Homological algebra exercise sheet Week 8

1. Let $E_{\bullet,\bullet}$ be a first quadrant double complex with horizontal differentials d^h and vertical differentials d^v. Define the page E^0 as follows: for integers $p, q \geq 0$, let $E^0_{p,q} = E_{p,q}$ and $d^0_{p,q} = d^v_{p,g}$. Each column $E^0_{p,\bullet}$ forms a chain complex, so we may define $E^1_{p,q} = H_q(E^0_{p,\bullet})$ and let $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ be the map that $d^h_{p,q}$ induces on homology. Lastly, note that each row $E^1_{\bullet,q}$ forms a chain complex, so we may define $E^2_{p,q} = H_p(E^1_{\bullet,q})$. The purpose of this exercise is to calculate the differentials $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$.

   (a) Show that $E^2_{p,q}$ can be presented as the group of all pairs (a, b) in $E_{p-1,q+1} \times E_{p,q}$ such that $0 = d^v b = d^v a + d^h b$, modulo the relation that these pairs are trivial: $(a, 0)$; $(d^h x, d^v x)$ for $x \in E_{p,q+1}$; and $(0, d^h c)$ for all $c \in E_{p+1,q}$ with $d^v c = 0$.

   (b) If $d^h(a) = 0$, show that such a pair (a, b) determines an element of $H_{p+q}(T)$, where T is the total complex of the double complex $E_{\bullet,\bullet}$.

   (c) Show that the formula $d(a, b) = (0, d^h(a))$ determines a well-defined map
   $$d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}.$$

2. In this exercise, $E_{\bullet,\bullet}$ is still a first quadrant double complex, and E^0, E^1 and E^2 are the pages of the same spectral sequence described in the previous exercise. Let T be the total complex of the double complex $E_{\bullet,\bullet}$. By diagram chasing, show that $E^2_{0,0} = H_0(T)$, and that there is an exact sequence
   $$H_2(T) \to E^2_{2,0} \xrightarrow{d} E^2_{0,1} \to H_1(T) \to E^2_{1,0} \to 0.$$

3. Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $q = 0, 1$. Show that there is a long exact sequence
   $$\cdots \to H_{p+1} \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p,0} \xrightarrow{d} E^2_{p-2,1} \to H_{p-1} \to \ldots$$

4. (Mapping Lemma for E^∞) Let $f : \left\{E^r_{pq}\right\} \to \left\{E'^r_{pq}\right\}$ be a morphism of spectral sequences such that for some r, $f^r : E^r_{pq} \cong E'^r_{pq}$ is an isomorphism for all p and q. Show that $f^\infty_{pq} : E^\infty_{pq} \cong E'^\infty_{pq}$ as well.